A MONTE CARLO STUDY OF THE ACCURACY AND ROBUSTNESS OF TEN BIVARIATE LOCATION ESTIMATORS

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Abstract. In a Monte Carlo study, ten bivariate location estimators are compared as regards their accuracy and robustness. In addition to the arithmetic mean, five bivariate medians and four depth-based trimmed means are thus investigated. The behavior of the estimators is examined under various sampling situations determined by three sample sizes and twenty-six underlying distributions, fourteen of which are centrally symmetric and twelve are asymmetric contaminated normals. Performance is assessed through numerical functions of the sample mean squared error and bias matrices.

RUNNING TITLE: Accuracy and Robustness of Location Estimators

1. Introduction

Robust alternatives to the arithmetic mean for estimating location have a history going back at least to Laplace [see Stigler (1986, p. 54)]. Fisher (1922) drew attention to the inefficiency of the arithmetic mean as an estimator of location for some distributions belonging to the family of Pearson curves near the normal. Using his normal contamination models, Tukey (1960) dramatically demonstrated how little efficient the mean can become when contamination increases. The same paper also shows how alternative location estimators such as the median or trimmed means can achieve higher asymptotic efficiency than the mean. As a result, statisticians have become more wary of making uncritical use of normal theory and have felt aware of the need for robust procedures, in the sense of procedures that remain good when the assumed model does not quite fit.

A theory of robust estimation was first developed by Huber (1964) in a paper that also introduced the so-called M-estimators of location, a class of estimators that includes the arithmetic mean, the median and maximum likelihood estimators. Huber defined a robust estimator of location as one that minimizes the asymptotic

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variance over some neighborhood of a known distribution (such as the normal). In the wake of Huber's work, several statisticians assessed the robustness and efficiency of various classes of location estimators for a variety of parent distributions; these classes include estimators defined by linear combination of order statistics (L-estimators) and estimators defined by rank tests (R-estimators). This effort culminated in a major Monte Carlo robustness study conducted at Princeton by Andrews et al. (1972), in which the performances of 68 univariate estimators of location were compared on samples from a dozen distributions. For good historical accounts of the development of these ideas, see Huber (1972) and Hampel et al. (1986).

Bickel (1964) was one of the first to study efficiency and robustness for multivariate location estimators. In his paper, the vector of coordinate medians and the vector of coordinate medians of averages of pairs (vector of coordinate Hodges-Lehmann estimators) are compared with the mean with respect to asymptotic efficiency under various distributions, some close to the normal, some not; in addition, Bickel investigates the robustness of the vector of coordinate Hodges-Lehmann estimators when the underlying distribution is contaminated.

Beginning in the seventies, a few authors obtained several multivariate versions of typically univariate notions such as medians, L-estimators, R-estimators. Four of the multivariate medians are now known as the spatial median (also called mediancenter or L_1-median), the Tukey or halfspace median, the Oja median and the Liu or simplicial median; these estimators were respectively proposed or discussed by Gini and Galvani (1929) [see also Haldane (1948)], Tukey (1975), Oja (1983) and Liu (1990). Some work has been done on the efficiency of the spatial median and the Oja median with respect to the arithmetical mean; a good reference for some of the results is Hettmansperger and McKean (1998).

Through his notion of depth, Tukey (1975) initiated a very fruitful approach for defining multivariate location estimators. A depth can be seen as a device for measuring the centrality of a multivariate data point within a given data cloud. Each such function induces a center-outward ranking of data points within a given multivariate data set, thus allowing a multivariate generalization of univariate location estimators such as the median, trimmed means or, more generally, L-estimators of location. In particular, Tukey's depth is the basis for the so-called halfspace median and a notion of trimmed mean used in this paper. Various depth functions have since appeared in the statistical literature [see Zuo and Serfling (2000a)], each one giving rise to a ranking of the data and therefore to a family of L-estimators of location. Besides Tukey's depth, the best known depth function is the simplicial depth of Liu (1990), which is used in this paper through the Liu median and Liu depth-based trimmed means.

In order that a depth be a useful applicable tool, ease of computation is a prerequisite. An algorithm was proposed by Niinimaa et al. (1992) to compute the Oja bivariate median. Until recently, no algorithms were available to compute the halfspace and simplicial depths, thus severely limiting the applicability of these functions. An important advance came with Rousseeuw and Ruts (1996) who constructed exact algorithms for computing the halfspace and simplicial depths of a point in a two-dimensional cloud. Rousseeuw and Ruts (1998) did the same for the computation of the bivariate Tukey median. Exact or approximate algorithms have also been obtained for higher dimensions: see Rousseeuw and Ruts (1998) and Struyf and Rousseeuw (2000).

Taking advantage of the algorithms recently proposed to compute depths, this Monte Carlo study aims at making a finite-sample comparison of the accuracy and robustness of ten bivariate estimators of location, six of which being based on Tukey's or Liu's depths. For each estimator, sample size and underlying distribution, performance is assessed through numerical functions of the estimated mean squared error and bias.

Little is known about the efficiency of the Tukey or Liu depths-based location estimators studied in this paper. Through a small simulation, Rousseeuw and Ruts (1998) have studied empirically the efficiency of the Tukey median and vector of medians for various sample sizes and the standard bivariate normal as the underlying distribution. Fraiman and Meloche (1999) have performed a similar study of six location estimators (including the vector of medians, as well as the spatial and Liu medians) for sample size 20 under various distributions, mostly different from ours. At the present time, the Monte Carlo method appears to be the only practical means of studying the efficiency of most depth-based location estimators. Indeed, for six of the depth-based location estimators chosen for this study, exact efficiency calculations remain intractable even for a normal distribution.

The paper is arranged as follows. Section 2 describes the location estimators studied in the simulation. Section 3 deals with the bivariate distributions which along with the sample size determine the sampling situations applied to the estimators. Section 4 covers some technical aspects of the Monte Carlo study and describes the numerical measures used to assess the estimators. Section 5 reports on the performances of the estimators and interprets the results. Section 6 is a conclusion.

2. Bivariate location estimators selected for the study

Let F be a probability distribution in $\mathbb{R}^2$ and $X_1, X_2, \ldots, X_n$ a random sample from F. A bivariate location estimator can be informally described as a $\mathbb{R}^2$-valued function T_n, defined for each sample size n, mapping the set of data points into some point $T_n(X_1, \ldots, X_n)$, which we understand as some approximation of the location or center of F.

In this Monte Carlo study, it will always be assumed that 0 *is the true location to be estimated.* This is the natural center for fourteen simulated distributions which are centrally symmetric about 0, but it is not so for twelve simulated distributions which are asymmetric contaminated distributions. Here we adopt the point of view that contamination is an uncontrollable disturbance in the estimation process that makes it more difficult to estimate correctly the true center 0, a target which one hopes ideally to estimate in the absence of contamination. This part of the study thus intends to examine the effect of some forms of contamination on the various estimators, as measured by bias and variability with respect to 0. Note that in case of asymmetry one may argue that several location parameters or natural centers are possible, all equally valid [see for example Lehmann (1983, p. 365)]; in such situation, one can in fact say that each location estimator has its own target (population mean, population trimmed mean, population Tukey median, etc.). In this study, it will be seen that asymmetric contamination has the expected effect of introducing a stronger bias component for each location estimator.

All estimators retained for the study satisfy some equivariance property. A location estimator T_n is said to be *translation equivariant* if $T_n(X_1+a, \ldots, X_n+a) = T_n(X_1, \ldots, X_n) + a$ for every $a \in \mathbb{R}^2$. Such an estimator is said to be *orthogonally equivariant* if $T_n(AX_1 + b, \ldots, AX_n + b) = AT_n(X_1, \ldots, X_n) + b$ holds for every linear orthogonal transformation $A : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and every $b \in \mathbb{R}^2$. If the latter equality holds for every nonsingular linear transformation, T_n is said to be *affine equivariant.*

Next we describe the ten bivariate location estimators that were investigated in the study.

2.1. **Affine equivariant estimators.**

2.1.1. *The mean.* This is the standard arithmetic mean $\sum_i X_i/n$.

2.1.2. *The Tukey median.* Let $U := \{u \in \mathbb{R}^2 : |u| = 1\}$. For any $x \in \mathbb{R}^2$ and $u \in U$, define the closed halfspace $H[x, u] := \{y \in \mathbb{R}^2 : u'y \geq u'x\}$. The *Tukey depth* (or *halfspace depth*) of a point $x \in \mathbb{R}^2$ (with respect to F) is defined as

$$HD(x) := \inf_{u \in U} FH[x, u].$$

The *empirical Tukey depth* HD_n is defined to be the Tukey depth with respect to the empirical distribution function F_n; it follows that $HD_n(x)$ is the minimal proportion of data points within a closed half-space containing x. Clearly the empirical Tukey depth attains a maximum value over $\mathbb{R}^2$; any maximizer is then called a Tukey median. Such maximizer is however not necessarily unique. Indeed, since HD_n is a quasi-concave function [Rousseeuw and Ruts (1999), Prop. 1], the set

$$K_n := \{y : HD_n(y) = \max_x HD_n(x)\}$$

is a non-empty closed convex polygon with a finite number of sides and vertices. In the following, to determine a maximizer uniquely, we agree to define it as the centroid of K_n [see Donoho and Gasko (1992), p. 1809]. This maximizer T_n is then said to be the *Tukey median* of the sample.

2.1.3. *The Liu median.* The *Liu depth* (or *simplicial depth*) of a point $x \in \mathbb{R}^2$ is defined to be the probability that x belongs to the simplex whose vertices are 3 independent observations from F. This can be written as

$$SD(x) := \int 1(x \in S(y_1, y_2, y_3))dF^3(y_1, y_2, y_3),$$

where $S(y_1, y_2, y_3)$ is the closed 2-dimensional simplex (triangle) with vertices y_1, y_2, y_3.

The sample version of SD is obtained by replacing F by F_n in the above, that is by computing the proportion of sample random triangles containing x:

$$SD_n(x) := \binom{n}{3}^{-1} \sum 1(x \in S(X_{i_1}, X_{i_2}, X_{i_3})),$$

where the sum ranges over all triplets i_1, i_2, i_3 such that $1 \le i_1 < i_2 < i_3 \le n$. As in Liu (1990), define a location estimator T_n as the data point where SD_n is maximized or the average of such points if there are many. We call T_n the *Liu median* of the sample.

2.1.4. *The Oja median.* Let $\Delta(x_1, x_2, x_3)$ denote the area of the triangle $S(x_1, x_2, x_3)$. Oja (1983) defined an *Oja median* as a point minimizing in x the function

$$\binom{n}{2}^{-1} \sum \Delta(X_{i_1}, X_{i_2}, x),$$

where the sum ranges over all pairs (i_1, i_2) such that $1 \le i_1 < i_2 \le n$. According to Oja and Niinimaa (1985), such a minimum is unique if n is even. In general, since the set of minimizers is a convex polygon [León and Massé (1993)], we agree to define the Oja median T_n as the centroid of this polygon.

2.1.5. *Depth-based trimmed means.* Depth functions provide a convenient way to rank data points according to their depth, and so enable us to define multivariate L-estimators of location. Here, to define trimmed means, we shall assume as in Liu et al. (1999) that the data points are ordered according to decreasing depth values. To define precisely r_i, the depth rank of X_i, we need to remove the ambiguity caused by the presence of ties: if $D_n(X_i) = D_n(X_j)$ and $i < j$, we agree that $r_i < r_j$. Write then $X_{[1]}, \dots, X_{[n]}$ for the corresponding (depth) order statistics.

Let Ave denote the average computed over the set of X_i's left after trimming according to some specification. In this paper, four trimmed means have been considered. For $\alpha = .05, .10$, these are:

$$T(F_n) = \text{Ave}(X_{[i]} : i \le n - \lfloor n\alpha \rfloor),$$

where both the empirical Tukey and Liu depths are used to determine the ordering.

2.2. **Two translation equivariant estimators that are not affine equivariant.**

2.2.1. *The vector of coordinate medians.* Writing $X_i = (X_{i1}, X_{i2})$, the vector of coordinate medians is defined as $T_n := (\mathrm{Med}(X_{11}, \dots, X_{n1}), \mathrm{Med}(X_{12}, \dots, X_{n2}))$, where Med denotes the one-dimensional median. This location estimator is easily seen to be translation equivariant but not affine equivariant.

2.2.2. *The spatial median.* The spatial median is defined as

$$T_n := \operatorname*{argmin}_x \frac{\sum_i |x - X_i|}{n},$$

where $|\cdot|$ is the euclidean norm. Except for degenerate cases, the foregoing minimum is known to be unique. Even though it is not affine equivariant, the spatial median is clearly translation equivariant and orthogonally equivariant. In our study, the spatial median will prove to be the best overall of all ten location estimators. For some background and properties, refer to Small (1990); more recent results and references can also be found in Chaudhuri (1996) or Chakraborty and Chaudhuri (1999).

In our description and interpretation of the results (Section 5), the following descriptive abbreviations are used to identify the estimators:

- *cmed* (spatial median);
- *tmed* (Tukey median);
- *omed* (Oja median);
- *lmed* (Liu median);
- *vmed* (vector of coordinate medians);
- *mean* (arithmetic mean);
- hd_α (trimmed mean with respect to the Tukey depth, $\alpha = .05, .10$);
- sd_α (trimmed mean with respect to the Liu depth, $\alpha = .05, .10$).

3. Bivariate distributions simulated

Twenty-six distributions were investigated. Among these, fourteen are centrally symmetric about 0, where central symmetry about 0 means that X and $-X$ have the same distribution. The twelve remaining simulated distributions are asymmetric contaminated normal distributions. One of the distributions has finite support and the others have been chosen such that heavy-tailedness ranges from low to high.

3.1. **Non mixtures centrally symmetric distributions.** These are:

1. $D_1 \equiv N_2(0, I)$, the standard bivariate normal distribution with density
$$f(x, y) = \frac{1}{2\pi} \exp(-(x^2 + y^2)/2));$$
2. $D_2 \equiv DE_2(0, I)$, the double exponential or Laplace bivariate distribution with independent components, whose density is
$$f(x, y) = \frac{1}{4} \exp\{-(|x| + |y|)\};$$
3. $D_3 \equiv UD$, the uniform distribution on the disk $\{(x, y) : 0 \leq x^2 + y^2 \leq 4\}$;
4. $D_4 \equiv t_3$, the t bivariate distribution with 3 degrees of freedom whose density is
$$f(x, y) = \frac{1}{2\pi} \left(1 + \frac{x^2 + y^2}{3}\right)^{-5/2};$$
5. $D_5 \equiv C_2(0, I)$, the standard bivariate Cauchy distribution with density
$$f(x, y) = \frac{1}{2\pi} \frac{1}{(1 + x^2 + y^2)^{3/2}}.$$

Note that for distributions D_2 and D_3 the scale has been chosen such that marginal distributions have unit variance.

3.2. **Centrally symmetric contaminated normals.** Nine simulated distributions are two-component mixtures with the main component being the standard bivariate normal distribution and the contamination component being a distribution centrally symmetric about 0. For $\alpha = .05, .10$ or $.20$, these are:

1. $MN(\alpha) := (1-\alpha)N_2(0, I) + \alpha N_2(0, \Sigma)$, $\Sigma = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$, respectively denoted D_6, D_7, D_8;
2. $MNC(\alpha) := (1 - \alpha)N_2(0, I) + \alpha C_2(0, I)$, respectively denoted D_9, D_{10}, D_{11};
3. $MNU(\alpha) := (1 - \alpha)N_2(0, I) + \alpha \begin{pmatrix} Z_1/U_1 \\ Z_2/U_2 \end{pmatrix}$, where $(Z_1, Z_2)'$ and $(U_1, U_2)'$ are independent, $(Z_1, Z_2)'$ is $N_2(0, I)$, and U_1, U_2 are independent uniform on $(0, 1)$, these mixtures being respectively denoted D_{12}, D_{13}, D_{14}.

Among the fourteen centrally symmetric distributions, D_4 to D_{14} may be described as being heavy-tailed to various degrees.

3.3. **Asymmetric contaminated normals.** Twelve simulated distributions are two-component mixtures with the main component being the standard bivariate normal distribution and the contamination component being a normal distribution centered at a point $\neq 0$. For $\alpha = .05, .10$ or $.20$, these are:

1. $MN_{11}(\alpha) := (1-\alpha)N_2(0,I) + \alpha N_2(\mu_1,\Sigma_1)$, where $\mu_1 = (2,2)'$ and $\Sigma_1 = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$, respectively denoted D_{15}, D_{16}, D_{17};
2. $MN_{12}(\alpha) := (1-\alpha)N_2(0,I) + \alpha N_2(\mu_1,\Sigma_2)$, where $\mu_1 := (2,2)'$ and $\Sigma_2 = \begin{pmatrix} 9 & 4 \\ 4 & 4 \end{pmatrix}$, respectively denoted D_{18}, D_{19}, D_{20};
3. $MN_{21}(\alpha) := (1-\alpha)N_2(0,I) + \alpha N_2(\mu_2,\Sigma_1)$, where $\mu_2 = (5,5)'$ and $\Sigma_1 = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$, respectively denoted D_{21}, D_{22}, D_{23};
4. $MN_{22}(\alpha) := (1-\alpha)N_2(0,I) + \alpha N_2(\mu_2,\Sigma_2)$, where $\mu_2 = (5,5)'$ and $\Sigma_2 = \begin{pmatrix} 9 & 4 \\ 4 & 4 \end{pmatrix}$, respectively denoted D_{24}, D_{25}, D_{26}.

4. Description of the Monte Carlo study

4.1. **Sample sizes and number of replications.** In what follows, each combination of distribution and sample size is called a sampling situation. This Monte Carlo experiment studies the behavior of the ten bivariate location estimators under 78 sampling situations determined by three sample sizes: 20, 60 and 200, and twenty-six distributions. For each sampling situation, 500 replications were used to take into account sampling variability.

4.2. **Algorithms.** The Ranlib library of Fortran routines for random number generation has been used. Johnson (1987) provides algorithms for the simulation of distributions that are ingredients of the twenty-six distributions used in this paper.

The algorithm of Rousseeuw and Ruts (1996) was used to compute both the Tukey depth and the Liu depth of a point. The Tukey median was obtained from the algorithm in Rousseeuw and Ruts (1998). The calculation of the spatial median was based on algorithms AS 78 (Gower, 1974) and AS 143 (Bedall and Zimmermann, 1979); that of the Oja median was based on algorithm AS 277 (Niinimaa et al.,1992). The simulation was done on seven SUN Ultra 5 workstations and the total execution time was about 260 hours (10 hours per distribution).

In the course of the simulation, the algorithms to compute the Tukey and Oja medians needed some adjustments. For some datasets, the algorithm in Rousseeuw and Ruts (1998) makes use of "dithering" to compute the Tukey median. In some cases where extreme observations were present, this yielded a "median" that was outside the convex envelope of the sample. To remedy this problem, a slower method of calculation has been implemented in which depth was computed on points of a grid centered on some preliminary estimate of the center. Since it proved to be unable to handle some exceptional datasets when $n = 200$, the algorithm by Niinimaa et al. (1992) to compute the Oja median also had to be slightly modified. Two S-Plus programs using Fortran code and implementing the modifications can be obtained from the authors.

4.3. **Measures of performance.** Given a bivariate point estimator T_n of $\theta \in \mathbb{R}^2$, the following identity is readily obtained:

$$E(T_n - \theta)(T_n - \theta)' = E(T_n T_n') - E(T_n)E(T_n)' + (\theta - E(T_n))(\theta - E(T_n))', \tag{1}$$

provided the expectations are well defined. By analogy with the well known property of real-valued estimators, the above can be written

$$MSE(T_n) = \mathrm{Var}(T_n) + \mathrm{SQB}(T_n),$$

where $MSE(T_n)$, $\mathrm{Var}(T_n)$ and $\mathrm{SQB}(T_n)$ will be called respectively the *mean squared error matrix*, the *covariance matrix* and the *squared bias matrix* of T_n. For each sampling situation in this study, we assume, as noted at the beginning of Section 2, that all location estimators T_n aim at estimating the true location parameter $\theta = 0$.

Let T_n^i denote the value of T_n for the i-th replication, $i = 1, \dots, 500$. Then,

$$S_n := \frac{1}{500} \sum_i (T_n^i - \overline{T}_n)(T_n^i - \overline{T}_n)'$$

and

$$\overline{T}_n := \frac{1}{500} \sum_i T_n^i$$

are respectively the sample covariance matrix and sample bias of T_n. At the sampling level, the above identity says that

$$\frac{1}{500} \sum_i T_n^i T_n^{i'} = S_n + \overline{T}_n \overline{T}_n',$$

which we will write as

$$\widehat{MSE}(T_n) = S_n + \widehat{\mathrm{SQB}}(T_n). \tag{2}$$

Note that (2) holds whether or not the expectations are defined in (1).

The performance of each location estimator will be assessed by numerical functions of its sample mean squared error and squared bias matrices. For several estimators, $n\mathrm{Var}(T_n) \to \Sigma$ and $n\,\mathrm{SQB}(T_n) \to 0$ as $n \to \infty$. For these estimators, assessing sample mean squared error, variability and bias is therefore less dependent upon the sample sizes if one uses the scaled matrices $n\widehat{MSE}(T_n)$, nS_n and $n\widehat{\mathrm{SQB}}(T_n)$. In the following, two numerical measures of accuracy are computed for each estimator and each sampling situation. The first of these measures is a function of the sample mean squared error matrix: $M := |n\widehat{MSE}(T_n)|^{1/2} = n|\widehat{MSE}(T_n)|^{1/2}$, where $|\cdot|$ denotes the determinant. (Through the covariance matrix component of the mean squared error matrix, M is also a measure of variability.) The second measure of accuracy is a function of the sample squared bias matrix: $B := [\mathrm{tr}(n\widehat{\mathrm{SQB}}(T_n))]^{1/2} = n^{1/2}[\mathrm{tr}(\widehat{\mathrm{SQB}}(T_n))]^{1/2}$, where tr is the trace function. In the next section, robustness will also be measured through M and B after restricting their use to the heavy-tailed distributions D_4 to D_{26}.

Recall that the determinant of a covariance matrix is sometimes called the generalized variance. Use of square roots is motivated here by the fact that, for bivariate asymptotically normal bivariate estimators, asymptotic relative efficiency may be defined as the ratio of the square roots of the generalized variances of the asymptotic normal distributions [see Bickel (1964, p. 1083) or Serfling (1980, p. 141)].

5. Results and interpretation

The results are presented in a series of five tables at the end of the paper. In all those tables, the performances of the location estimators are compared through the numerical measures M and B, or some function of these.

5.1. Assessment of accuracy.

As explained above, both M and B are viewed as measures of accuracy. Small values of M reflect accuracy through low variability and/or low bias; large values imply high variability and/or bias. From this point of view, a highly accurate (resp. inaccurate) estimator is one having simultaneously low (resp. high) values of M and B. Obviously, the only aspect of accuracy measured by B is bias.

Accuracy is first assessed through Table 1. In the latter, for each sampling situation, the estimators are ranked according to increasing values of M (first and second lines) and B (third and fourth lines). Table 2 is then used to assess the accuracy of each estimator with respect to M, independently of the sample size and for each sampled distribution. To construct that table, estimators are first ordered according to increasing values of M for each sample size; for each estimator, the corresponding midranks are then averaged over all three samples sizes. Finally, accuracy is also assessed through Table 3, which is the counterpart of Table 2 for B. For both tables, estimators have been ordered according to increasing values of the average midrank over all distributions, denoted $\overline{r}_M$ for Table 2 and $\overline{r}_B$ for Table 3.

As expected from well-known optimality properties, for all sample sizes *mean* is the most accurate estimator for $N_2(0, I)$ and UD. Unsurprisingly, the same estimator turns up as extremely inaccurate for distributions $C_2(0, I)$, $MNC(\alpha)$ and $MNU(\alpha)$, all having no expectation. *mean* also exhibits average to poor accuracy for $DE_2(0, I)$, t_3, as well as mixtures $MN(\alpha)$ and $MN_{ij}(\alpha)$ (except for $MN_{11}(.05), n = 20$), the more so as the sample size grows. In all cases of asymmetric contamination, Table 3 shows that *mean* has overall the strongest bias.

It is seen that the 5%-trimmed means $hd_{.05}$ and $sd_{.05}$ coincide when $n = 20, 60$, and differ very slightly when $n = 200$. For all sizes, their accuracy ranges from fairly good to very good for $N_2(0, I)$, UD and the mixtures $MN(.05)$ and $MN(.10)$; for $n = 200$, they tend to be very good for $MNC(.05)$, $MNC(.10)$ and $MNU(.05)$. These estimators perform rather poorly for $DE_2(0, I)$, t_3, $C_2(0, I)$, for most mixtures when $n = 20, 60$ and practically all cases of asymmetric contamination. As seen from Table 3, a strong bias is present for $C_2(0, I)$ and all mixtures.

The 10%-trimmed means $hd_{.10}$ and $sd_{.10}$ are identical when $n = 20$ and, except for the $MNU(\alpha)$'s, not significantly different when $n = 60$ or 200. Their accuracy ranges from good to very good for all sample sizes and distributions $N_2(0, I)$, $DE_2(0, I)$, UD, t_3, all $MN(\alpha)$'s; if $n = 200$, the same holds true for all centrally symmetric mixtures. For all sizes, $hd_{.10}$ and $sd_{.10}$ are very inaccurate for $C_2(0, I)$ and most asymmetric contaminated normals; if $n = 20$, they perform poorly for several symmetric mixtures. It is also noted that these estimators are strongly biased for asymmetric contaminated normals, though much less so than the 5%-trimmed means. For heavy-tailed distributions $D_4 - D_{26}$ and $n = 60, 200$, the 10%-trimmed means tend to be more accurate than the 5%-trimmed means.

The study shows that for all sample sizes and distributions, *lmed* estimates the center rather poorly. Its performance ranges from average for heavy-tailed distributions such as $C_2(0, I), MNC(.20), MNU(.20), MN_{12}(.20), MN_{21}(.20), MN_{22}(.20)$, to very inaccurate for mixtures $MN(\alpha), MN_{11}(.05), MN_{11}(.10)$ and $MN_{12}(.05)$. As size n grows, *lmed* tends to improve for distributions $MN_{11}(.20), MN_{12}(.10)$, $MN_{22}(.05)$, while it tends to worsen for distributions $t_3, MNC(\alpha)$ and $MNU(\alpha)$. It is also noted that from Table 3 that *lmed* is strongly biased for $N_1(0, I), UD$ and t_3. This poor performance of *lmed* supports the conclusions of a simulation by Fraiman and Meloche (1999) for a different set of distributions.

It is seen that *vmed* is a very accurate estimator of the center for $DE_2(0, I)$, as expected from optimality properties of the usual median. For distributions $N_2(0, I)$, UD, $MN(.05)$ and $MN_{11}(.05)$, *vmed* has a rather poor performance. For all other distributions, the accuracy of *vmed* is fairly good, either measured from M or from B.

For smaller sample sizes ($n = 20, 60$), *tmed* is generally among the top three most accurate estimators for heavy-tailed distributions such as t_3 and $C_2(0, I)$, as well as all mixtures. It does not however perform very well for most mixtures when $n = 200$, and also for $N_2(0, I)$ and UD for all sample sizes. The performance of *tmed* thus appears to depend on the sample size, more so than for other estimators. For $N_2(0, I)$, the results of this study agree with those of Rousseeuw and Ruts (1998).

For most sampling situations considered in the study, *omed* tends to be the second most accurate estimator. A few exceptions are worth noting: for distributions $N_2(0, I)$ and UD, the performance is rather mediocre; for $MN(.05)$ and $MNC(.05)$, it is average. It is interesting to see that *omed* is more accurate for $n = 200$ than for sizes $n = 20, 60$.

Finally, Tables 2 and 3 show that overall *cmed* is the most accurate and least biased estimator. One can see from Table 1 that this tends to hold for most sampling situations. Distributions for which more accurate and less biased estimators can be found are $N_2(0, I), DE_2(0, I), UD$ and some symmetric mixtures.

5.2. **Assessment of robustness.** Tables 4 and 5 allow us to compare the ten location estimators according to a score of robustness which we now define for Table 4. Given an estimator and a sample size, the score of that estimator is obtained from Table 1 by computing the midrank of that estimator according to increasing values of M, then by taking the average of those midranks over the 23 heavy-tailed distributions D_4 to D_{26}. Similarly, scores in Table 5 are obtained using B as the measure of accuracy. For each of these tables, the last line also shows the average score of robustness $\overline{s}$ over the three sample sizes.

It is noted that in estimating the center of heavy-tailed distributions some estimators are more dependent on the sample size than others. The medians *cmed*, *omed*, *vmed*, *lmed* show little dependence, in contrast with the means and *tmed* which all exhibit some degree of dependence. For *tmed*, this phenomenon may be due to instability of the algorithm used to compute it when sample size is large. In case of the trimmed means, the score tends to get smaller when the sample size grows.

For sample sizes $n = 60$ or 200, $hd_{.10}$ and $sd_{.10}$ are by far the most robust of all means. Overall, these trimmed means are less robust than medians *cmed*, *omed*, *tmed* and *vmed*, but they appear as good if not better than *lmed*. As expected, *mean* is the least robust of all estimators, showing the worst performance when $n = 200$.

With respect to the average score of robustness, whether it is measured from M or B, *cmed* clearly outperforms its competitors, with *omed*, *tmed* and *vmed* appearing second, third and fourth. On the whole the least robust of all estimators is *lmed*. Two medians, *cmed* and *omed*, show strong robustness for all sample sizes.

6. Conclusion

The performance of ten bivariate location estimators has been investigated under seventy-eight sampling situations, the primary concern always being accuracy and robustness. In addition to three sample sizes, fourteen centrally symmetric distributions and twelve asymmetric contaminated normals were retained for the study, most of these distributions having some degree of heavy-tailedness.

The study has shown that four bivariate medians and, to a lesser degree, two depth-based trimmed means are very good alternatives to the arithmetic mean for the estimation of a location parameter. Of all estimators, the spatial median clearly stands as the best overall, followed by the Oja median and the Tukey median. Furthermore, four of the medians are overall better than any of the depth-based trimmed means. The latter perform best when the degree of trimming is highest (in our case 10%) and when $n = 200$; all four trimmed means lack robustness when the sample size is small ($n = 20$).

Finally, it is noted that the good performance of the spatial median is not entirely explained with help of the well-known measure of robustness of finite-sample (replacement) breakdown point. Roughly speaking, this is the smallest fraction of the n observations which, appropriately modified, can send the values of the estimator arbitrarily far away. The spatial median and vector of coordinate medians are known to have the same breakdown point of $\lfloor (n+1)/2 \rfloor / n$, the highest possible value for a translation equivariant estimator [Lopuhaä and Rousseeuw (1991)]. Since the vector of coordinate medians is seen to rank fourth in this study, the superior performance of the spatial median thus appears to be largely due to its high efficiency, a quality reflected through low values of M and B.

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TABLE 1. Estimated accuracy as measured by M and B. For each distribution and each sample size, estimators ordered according to increasing M (first and second lines) and B (third and fourth lines).

$N_2(0,I)$	20	$mean$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$cmed$	$tmed$	$omed$	$vmed$	$lmed$
		0.969	1.077	1.077	1.164	1.164	1.206	1.218	1.236	1.382	2.189
		$mean$	$tmed$	$cmed$	$omed$	$vmed$	$lmed$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		0.007	0.018	0.024	0.025	0.029	0.036	0.038	0.038	0.048	0.048
	60	$mean$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$cmed$	$omed$	$tmed$	$vmed$	$lmed$
		0.984	1.108	1.108	1.113	1.116	1.266	1.286	1.323	1.467	2.184
		$mean$	$sd_{.10}$	$hd_{.10}$	$tmed$	$hd_{.05}$	$sd_{.05}$	$vmed$	$omed$	$cmed$	$lmed$
		0.095	0.099	0.100	0.105	0.115	0.115	0.116	0.118	0.118	0.156
	200	$mean$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	$sd_{.10}$	$omed$	$cmed$	$tmed$	$vmed$	$lmed$
		1.069	1.084	1.087	1.087	1.158	1.331	1.332	1.363	1.591	2.426
		$hd_{.05}$	$sd_{.05}$	$mean$	$hd_{.10}$	$lmed$	$sd_{.10}$	$tmed$	$omed$	$cmed$	$vmed$
		0.045	0.046	0.053	0.055	0.056	0.057	0.067	0.075	0.075	0.095
$DE_2(0,I)$	20	$vmed$	$cmed$	$omed$	$tmed$	$mean$	$hd_{.05}$	$sd_{.05}$	$lmed$	$hd_{.10}$	$sd_{.10}$
		0.635	0.696	0.712	0.735	0.929	1.068	1.068	1.136	1.169	1.169
		$cmed$	$omed$	$vmed$	$tmed$	$mean$	$hd_{.05}$	$hd_{.10}$	$sd_{.05}$	$sd_{.10}$	$lmed$
		0.017	0.022	0.027	0.034	0.043	0.046	0.046	0.046	0.046	0.066
	60	$vmed$	$omed$	$tmed$	$cmed$	$hd_{.10}$	$sd_{.10}$	$mean$	$lmed$	$hd_{.05}$	$sd_{.05}$
		0.608	0.671	0.697	0.699	1.013	1.015	1.027	1.065	1.136	1.136
		$lmed$	$omed$	$vmed$	$cmed$	$tmed$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$mean$
		0.034	0.037	0.037	0.039	0.047	0.065	0.065	0.069	0.076	0.080
	200	$vmed$	$omed$	$tmed$	$cmed$	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	$mean$	$lmed$
		0.503	0.594	0.621	0.636	0.898	0.932	0.937	0.944	0.958	0.981
		$lmed$	$sd_{.05}$	$omed$	$vmed$	$cmed$	$tmed$	$sd_{.10}$	$hd_{.05}$	$hd_{.10}$	$mean$
		0.079	0.081	0.085	0.087	0.088	0.091	0.093	0.098	0.100	0.101
UD	20	$mean$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$cmed$	$tmed$	$omed$	$vmed$	$lmed$
		1.035	1.105	1.105	1.186	1.186	2.053	2.058	2.128	2.267	6.670
		$lmed$	$mean$	$tmed$	$omed$	$cmed$	$hd_{.05}$	$sd_{.05}$	$vmed$	$hd_{.10}$	$sd_{.10}$
		0.041	0.085	0.093	0.101	0.105	0.108	0.108	0.109	0.128	0.128
	60	$mean$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$cmed$	$omed$	$tmed$	$vmed$	$lmed$
		0.968	1.080	1.080	1.179	1.179	1.921	1.963	2.006	2.389	37.36
		$mean$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$omed$	$cmed$	$vmed$	$tmed$	$lmed$
		0.045	0.053	0.053	0.066	0.066	0.076	0.081	0.083	0.091	0.341
	200	$mean$	$hd_{.05}$	$sd_{.05}$	$sd_{.10}$	$hd_{.10}$	$omed$	$cmed$	$tmed$	$vmed$	$lmed$
		1.054	1.145	1.145	1.161	1.173	2.078	2.082	2.156	2.658	156.2
		$mean$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$tmed$	$omed$	$cmed$	$vmed$	$lmed$
		0.034	0.036	0.036	0.047	0.052	0.056	0.064	0.065	0.138	0.806
t_3	20	$cmed$	$tmed$	$omed$	$vmed$	$lmed$	$mean$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		1.504	1.557	1.594	1.725	2.520	3.135	3.412	3.412	3.442	3.442
		$vmed$	$tmed$	$cmed$	$omed$	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	$mean$	$lmed$
		0.047	0.075	0.080	0.090	0.096	0.096	0.110	0.110	0.113	0.153
	60	$cmed$	$tmed$	$omed$	$vmed$	$sd_{.10}$	$hd_{.10}$	$lmed$	$hd_{.05}$	$sd_{.05}$	$mean$
		1.577	1.592	1.594	1.923	2.131	2.155	2.732	2.741	2.741	2.811
		$tmed$	$lmed$	$omed$	$cmed$	$hd_{.10}$	$vmed$	$hd_{.05}$	$sd_{.05}$	$sd_{.10}$	$mean$
		0.018	0.019	0.025	0.035	0.061	0.062	0.070	0.070	0.079	0.087
	200	$cmed$	$omed$	$vmed$	$hd_{.10}$	$sd_{.10}$	$tmed$	$hd_{.05}$	$sd_{.05}$	$lmed$	$mean$
		1.387	1.394	1.688	1.815	1.883	2.025	2.061	2.092	2.574	3.012
		$sd_{.10}$	$sd_{.05}$	$hd_{.10}$	$hd_{.05}$	$tmed$	$omed$	$cmed$	$vmed$	$mean$	$lmed$
		0.006	0.011	0.012	0.015	0.018	0.029	0.030	0.038	0.049	0.095

TABLE 1. (Continued) Estimated accuracy as measured by M and B. For each distribution and each sample size, estimators ordered according to increasing M (first and second lines) and B (third and fourth lines).

$C_2(0,1)$	20	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		2.512	2.639	2.795	2.843	3.754	7214	7214	7483	7782	7782
		omed	*tmed*	*cmed*	*vmed*	*lmed*	$hd_{.05}$	$sd_{.05}$	*mean*	$hd_{.10}$	$sd_{.10}$
		0.007	0.019	0.034	0.036	0.054	3.639	3.639	3.663	4.939	4.939
	60	*cmed*	*omed*	*tmed*	*vmed*	*lmed*	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		2.163	2.254	2.331	2.556	3.288	173.4	188.3	8196	8196	$> 10^4$
		vmed	*omed*	*cmed*	*tmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.113	0.114	0.136	0.140	0.257	1.770	1.909	7.141	7.141	11.43
	200	*cmed*	*omed*	*vmed*	*lmed*	*tmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		2.139	2.226	2.533	3.091	7.019	10.22	10.48	22.14	26.77	$> 10^5$
		vmed	*cmed*	*omed*	*lmed*	$hd_{.10}$	*tmed*	$hd_{.05}$	$sd_{.10}$	$sd_{.05}$	*mean*
		0.024	0.030	0.038	0.061	0.156	0.177	0.189	0.242	0.409	17.43
$MN(.05)$	20	*tmed*	*cmed*	*omed*	*mean*	$hd_{.05}$	$sd_{.05}$	*vmed*	$hd_{.10}$	$sd_{.10}$	*lmed*
		1.344	1.357	1.383	1.388	1.510	1.510	1.569	1.592	1.592	2.486
		vmed	*mean*	*lmed*	*cmed*	*omed*	*tmed*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		0.066	0.075	0.089	0.101	0.107	0.108	0.114	0.114	0.140	0.140
	60	*cmed*	*omed*	*tmed*	$sd_{.10}$	$hd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$	*vmed*	*lmed*
		1.323	1.327	1.342	1.352	1.359	1.401	1.513	1.513	1.604	2.376
		tmed	$hd_{.10}$	$sd_{.10}$	*omed*	*cmed*	*mean*	*vmed*	$hd_{.05}$	$sd_{.05}$	*lmed*
		0.018	0.019	0.020	0.024	0.031	0.058	0.059	0.064	0.064	0.120
	200	$sd_{.05}$	$hd_{.10}$	$hd_{.05}$	$sd_{.10}$	*cmed*	*omed*	*mean*	*vmed*	*tmed*	*lmed*
		1.237	1.238	1.240	1.291	1.320	1.328	1.354	1.616	1.666	2.237
		lmed	*tmed*	*vmed*	$sd_{.05}$	*mean*	$hd_{.10}$	*omed*	*cmed*	$sd_{.10}$	$hd_{.05}$
		0.016	0.049	0.055	0.064	0.069	0.069	0.072	0.075	0.079	0.090
$MN(.10)$	20	*cmed*	*tmed*	*omed*	*vmed*	*mean*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	*lmed*
		1.519	1.534	1.562	1.715	1.870	2.039	2.039	2.100	2.100	2.576
		mean	*cmed*	*vmed*	*omed*	$hd_{.05}$	$sd_{.05}$	*tmed*	$hd_{.10}$	$sd_{.10}$	*lmed*
		0.077	0.084	0.084	0.096	0.096	0.096	0.107	0.118	0.118	0.191
	60	*cmed*	*omed*	$sd_{.10}$	$hd_{.10}$	*tmed*	*mean*	$hd_{.05}$	$sd_{.05}$	*vmed*	*lmed*
		1.370	1.382	1.396	1.402	1.419	1.592	1.624	1.624	1.654	2.450
		vmed	*lmed*	*tmed*	*cmed*	*omed*	$sd_{.10}$	*mean*	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$
		0.012	0.015	0.017	0.017	0.019	0.024	0.025	0.029	0.034	0.034
	200	$hd_{.10}$	*omed*	*cmed*	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	*tmed*	*vmed*	*mean*	*lmed*
		1.381	1.426	1.426	1.462	1.495	1.508	1.667	1.723	1.826	2.450
		tmed	*omed*	*cmed*	*vmed*	$sd_{.10}$	$hd_{.05}$	$hd_{.10}$	*mean*	$sd_{.05}$	*lmed*
		0.020	0.024	0.025	0.041	0.041	0.052	0.055	0.066	0.068	0.097
$MN(.20)$	20	*cmed*	*tmed*	*omed*	*vmed*	*mean*	$hd_{.05}$	$sd_{.05}$	*lmed*	$hd_{.10}$	$sd_{.10}$
		1.648	1.681	1.732	1.892	2.374	2.543	2.543	2.631	2.682	2.682
		$hd_{.10}$	$sd_{.10}$	*lmed*	*omed*	*cmed*	$hd_{.05}$	$sd_{.05}$	*tmed*	*vmed*	*mean*
		0.046	0.046	0.052	0.063	0.063	0.070	0.070	0.074	0.091	0.097
	60	*cmed*	*omed*	$sd_{.10}$	$hd_{.10}$	*vmed*	*tmed*	*mean*	$hd_{.05}$	$sd_{.05}$	*lmed*
		1.781	1.831	2.092	2.099	2.170	2.176	2.525	2.672	2.672	3.003
		vmed	*cmed*	*omed*	*tmed*	$hd_{.10}$	*lmed*	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.010	0.013	0.018	0.037	0.037	0.040	0.043	0.045	0.045	0.046
	200	*cmed*	*omed*	*tmed*	$sd_{.10}$	$hd_{.10}$	*vmed*	$sd_{.05}$	$hd_{.05}$	*mean*	*lmed*
		1.736	1.746	1.780	1.864	1.894	1.984	2.275	2.311	2.621	2.941
		$sd_{.10}$	$hd_{.10}$	*mean*	*cmed*	*omed*	$hd_{.05}$	*tmed*	*vmed*	$sd_{.05}$	*lmed*
		0.082	0.117	0.119	0.146	0.150	0.152	0.154	0.156	0.156	0.175

Table 1. (Continued) Estimated accuracy as measured by M and B. For each distribution and each sample size, estimators ordered according to increasing M (first and second lines) and B (third and fourth lines).

$MNC(.05)$	20	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	*mean*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		1.255	1.294	1.298	1.398	2.119	707.8	784.3	784.3	872.0	872.0
		omed	*tmed*	*vmed*	*cmed*	*lmed*	*mean*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		0.052	0.059	0.059	0.066	0.068	1.444	1.577	1.577	1.697	1.697
	60	*cmed*	*omed*	*tmed*	*vmed*	$sd_{.10}$	$hd_{.10}$	*lmed*	$hd_{.05}$	$sd_{.05}$	*mean*
		1.343	1.365	1.382	1.652	1.694	1.695	2.307	1553	1553	2244
		$hd_{.10}$	$sd_{.10}$	*vmed*	*lmed*	*cmed*	*tmed*	*omed*	$hd_{.05}$	$sd_{.05}$	*mean*
		0.016	0.026	0.068	0.073	0.075	0.079	0.085	3.133	3.133	3.435
	200	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*cmed*	$sd_{.10}$	*omed*	*vmed*	*tmed*	*lmed*	*mean*
		1.127	1.160	1.162	1.216	1.227	1.243	1.459	2.074	2.202	600.3
		$sd_{.05}$	*vmed*	$sd_{.10}$	$hd_{.05}$	*cmed*	$hd_{.10}$	*omed*	*tmed*	*lmed*	*mean*
		0.105	0.113	0.119	0.123	0.125	0.129	0.133	0.179	0.184	2.026
$MNC(.10)$	20	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		1.363	1.417	1.425	1.592	2.203	25.16	25.16	433.2	478.8	478.8
		lmed	*tmed*	*omed*	*cmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		0.051	0.057	0.062	0.065	0.066	0.087	0.087	3.266	3.344	3.344
	60	*omed*	*cmed*	*tmed*	*vmed*	$sd_{.10}$	$hd_{.10}$	*lmed*	$hd_{.05}$	$sd_{.05}$	*mean*
		1.229	1.231	1.233	1.473	1.995	2.001	2.332	138.2	138.2	384.6
		omed	$sd_{.10}$	$hd_{.10}$	*tmed*	*vmed*	*lmed*	*cmed*	$hd_{.05}$	$sd_{.05}$	*mean*
		0.077	0.080	0.087	0.091	0.095	0.096	0.099	1.102	1.102	1.775
	200	*omed*	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	$sd_{.10}$	*vmed*	*tmed*	*lmed*	*cmed*	*mean*
		1.378	1.388	1.405	1.461	1.479	1.690	1.904	2.311	5.623	1005
		vmed	*tmed*	*omed*	$hd_{.05}$	$hd_{.10}$	$sd_{.05}$	$sd_{.10}$	*lmed*	*cmed*	*mean*
		0.047	0.064	0.077	0.087	0.093	0.095	0.105	0.109	0.114	0.855
$MNC(.20)$	20	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		1.421	1.430	1.453	1.683	2.356	49.94	49.94	3222	3439	3439
		tmed	*vmed*	*cmed*	*omed*	*lmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		0.093	0.095	0.096	0.120	0.161	0.911	0.911	18.22	19.17	19.17
	60	*cmed*	*omed*	*vmed*	*lmed*	$sd_{.10}$	$hd_{.10}$	*tmed*	$hd_{.05}$	$sd_{.05}$	*mean*
		1.305	1.367	1.660	2.247	2.343	2.425	5.015	49.66	49.66	$> 10^4$
		cmed	*omed*	*vmed*	$sd_{.10}$	*lmed*	$hd_{.10}$	*tmed*	$hd_{.05}$	$sd_{.05}$	*mean*
		0.042	0.054	0.057	0.119	0.129	0.145	0.231	0.548	0.548	51.90
	200	*cmed*	$hd_{.10}$	*omed*	$sd_{.10}$	*vmed*	$sd_{.05}$	$hd_{.05}$	*lmed*	*tmed*	*mean*
		1.367	1.576	1.577	1.613	1.637	1.790	1.889	2.390	23.56	$> 10^4$
		vmed	*cmed*	*omed*	$sd_{.05}$	$sd_{.10}$	*lmed*	$hd_{.10}$	$hd_{.05}$	*tmed*	*mean*
		0.008	0.038	0.041	0.048	0.060	0.068	0.071	0.079	0.808	28.01
$MNU(.05)$	20	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		1.380	1.398	1.437	1.536	2.264	33.47	33.47	37.37	38.89	38.89
		tmed	*omed*	*lmed*	*cmed*	*vmed*	*mean*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$
		0.070	0.071	0.073	0.075	0.108	0.777	0.797	0.797	0.874	0.874
	60	*cmed*	*omed*	*tmed*	*vmed*	*lmed*	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.384	1.394	1.402	1.611	2.234	29.26	29.53	243.2	243.2	245.9
		vmed	*cmed*	*omed*	*tmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.045	0.094	0.102	0.103	0.129	0.844	0.849	0.932	0.932	1.138
	200	$hd_{.10}$	$hd_{.05}$	*cmed*	*omed*	$sd_{.05}$	$sd_{.10}$	*vmed*	*tmed*	*lmed*	*mean*
		1.320	1.397	1.412	1.436	1.459	1.479	1.767	2.258	2.470	514.5
		cmed	*omed*	*vmed*	$hd_{.10}$	*lmed*	*tmed*	$hd_{.05}$	$sd_{.05}$	$sd_{.10}$	*mean*
		0.013	0.014	0.019	0.035	0.042	0.056	0.057	0.066	0.081	0.816

Table 1. (Continued) Estimated accuracy as measured by M and B. For each distribution and each sample size, estimators ordered according to increasing M (first and second lines) and B (third and fourth lines).

$MNU(.10)$	20	*tmed*	*cmed*	*omed*	*vmed*	*lmed*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	*mean*
		1.449	1.454	1.516	1.646	2.325	400.4	400.4	413.3	413.3	1013
		vmed	*tmed*	*cmed*	*omed*	*lmed*	*mean*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$
		0.090	0.093	0.099	0.106	0.112	0.221	1.179	1.179	1.437	1.437
	60	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.416	1.449	1.483	1.664	2.596	3.046	3.065	337.4	337.4	537.7
		vmed	$hd_{.10}$	$sd_{.10}$	*omed*	*tmed*	*cmed*	*lmed*	$hd_{.05}$	$sd_{.05}$	*mean*
		0.012	0.015	0.025	0.034	0.045	0.052	0.080	1.092	1.092	1.310
	200	*cmed*	*omed*	$hd_{.10}$	*vmed*	$hd_{.05}$	$sd_{.10}$	$sd_{.05}$	*lmed*	*tmed*	*mean*
		1.481	1.489	1.515	1.762	1.821	2.029	2.345	2.511	4.765	1801
		vmed	$hd_{.05}$	*omed*	$sd_{.05}$	$sd_{.10}$	*tmed*	*cmed*	$hd_{.10}$	*lmed*	*mean*
		0.025	0.032	0.038	0.041	0.045	0.051	0.054	0.054	0.069	5.759
$MNU(.20)$	20	*cmed*	*omed*	*vmed*	*lmed*	*tmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		1.691	1.779	1.864	2.788	72.58	263.1	263.1	402.9	421.6	421.6
		vmed	*lmed*	*omed*	*cmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*	*tmed*
		0.109	0.110	0.124	0.128	0.855	0.855	1.511	1.511	1.530	2.542
	60	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.717	1.773	1.821	2.109	2.750	3.741	6.528	6453	6453	8099
		tmed	*cmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$
		0.097	0.098	0.099	0.107	0.124	0.176	0.305	6.488	9.034	9.034
	200	*cmed*	*omed*	*vmed*	$hd_{.10}$	$sd_{.10}$	*lmed*	$hd_{.05}$	$sd_{.05}$	*tmed*	*mean*
		1.618	1.638	1.818	2.084	2.594	2.662	4.025	6.990	13.00	5199
		omed	*vmed*	*cmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	*lmed*	$sd_{.05}$	*tmed*	*mean*
		0.036	0.038	0.069	0.087	0.115	0.128	0.139	0.174	0.286	7.948
$MN_{11}(.05)$	20	*mean*	$hd_{.05}$	$sd_{.05}$	*cmed*	*tmed*	$hd_{.10}$	$sd_{.10}$	*omed*	*vmed*	*lmed*
		1.385	1.502	1.502	1.570	1.597	1.629	1.629	1.637	1.713	2.592
		cmed	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.251	0.259	0.261	0.287	0.305	0.535	0.535	0.541	0.541	0.572
	60	*cmed*	*omed*	*tmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$	*vmed*	*lmed*
		1.606	1.613	1.622	1.711	1.713	1.801	1.893	1.893	1.927	2.517
		lmed	*cmed*	*omed*	*tmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.581	0.584	0.585	0.589	0.621	0.941	0.942	1.019	1.019	1.094
	200	*cmed*	*omed*	*tmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*vmed*	*mean*	*lmed*
		1.741	1.747	1.789	2.057	2.062	2.152	2.164	2.233	2.354	2.847
		tmed	*cmed*	*omed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		0.928	0.933	0.941	0.968	1.053	1.610	1.611	1.707	1.723	1.999
$MN_{11}(.10)$	20	*cmed*	*omed*	*vmed*	*mean*	*tmed*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	*lmed*
		1.638	1.718	1.967	1.973	2.007	2.034	2.034	2.143	2.143	2.602
		cmed	*lmed*	*omed*	*tmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.519	0.554	0.554	0.599	0.634	1.078	1.078	1.131	1.131	1.187
	60	*cmed*	*omed*	*tmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	*mean*	$hd_{.05}$	$sd_{.05}$	*lmed*
		2.057	2.075	2.129	2.540	2.574	2.578	2.730	2.795	2.795	3.288
		cmed	*omed*	*tmed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.159	1.162	1.167	1.169	1.296	1.893	1.893	2.050	2.050	2.195
	200	*cmed*	*omed*	*tmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*lmed*	*mean*
		2.689	2.734	2.769	3.423	3.649	3.678	3.838	3.884	3.944	4.167
		cmed	*tmed*	*omed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		1.950	1.956	1.989	2.077	2.205	3.263	3.266	3.494	3.495	3.961

Table 1. (Continued) Estimated accuracy as measured by M and B. For each distribution and each sample size, estimators ordered according to increasing M (first and second lines) and B (third and fourth lines).

$MN_{11}(.20)$	20	*cmed*	*tmed*	*omed*	*vmed*	*mean*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	*lmed*
		2.428	2.621	2.652	2.991	3.203	3.350	3.350	3.576	3.576	3.791
		cmed	*lmed*	*tmed*	*omed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.367	1.402	1.427	1.453	1.574	2.359	2.359	2.450	2.450	2.514
	60	*cmed*	*omed*	*tmed*	*vmed*	$sd_{.10}$	$hd_{.10}$	*mean*	*lmed*	$hd_{.05}$	$sd_{.05}$
		3.218	3.381	3.413	4.133	4.682	4.693	4.741	4.782	4.978	4.978
		cmed	*tmed*	*omed*	*lmed*	*vmed*	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		2.296	2.386	2.397	2.447	2.645	3.848	3.855	4.063	4.063	4.232
	200	*cmed*	*omed*	*tmed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		5.214	5.415	5.542	7.175	7.380	7.728	7.877	8.057	8.075	8.394
		cmed	*lmed*	*tmed*	*omed*	*vmed*	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		4.400	4.461	4.563	4.570	5.037	7.115	7.163	7.511	7.531	8.049
$MN_{12}(.05)$	20	*cmed*	*omed*	*tmed*	*mean*	*vmed*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	*lmed*
		1.448	1.453	1.454	1.591	1.693	1.693	1.693	1.738	1.738	2.368
		cmed	*omed*	*tmed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.308	0.310	0.312	0.332	0.350	0.590	0.590	0.600	0.600	0.672
	60	*cmed*	*omed*	*tmed*	$hd_{.10}$	$sd_{.10}$	*vmed*	*mean*	$hd_{.05}$	$sd_{.05}$	*lmed*
		1.442	1.449	1.526	1.723	1.725	1.749	1.865	1.895	1.895	2.490
		vmed	*cmed*	*omed*	*tmed*	*lmed*	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.375	0.391	0.399	0.401	0.403	0.787	0.788	0.992	0.992	1.124
	200	*cmed*	*omed*	$hd_{.10}$	$sd_{.10}$	*vmed*	$hd_{.05}$	$sd_{.05}$	*tmed*	*mean*	*lmed*
		1.637	1.658	1.982	2.052	2.068	2.108	2.121	2.363	2.611	2.688
		lmed	*cmed*	*omed*	*tmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		0.727	0.800	0.813	0.876	0.916	1.476	1.478	1.575	1.593	2.142
$MN_{12}(.10)$	20	*tmed*	*cmed*	*omed*	*vmed*	*mean*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$	*lmed*
		1.665	1.667	1.674	1.981	2.191	2.291	2.291	2.390	2.390	2.724
		cmed	*tmed*	*omed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.429	0.440	0.442	0.473	0.510	0.999	0.999	1.136	1.136	1.195
	60	*cmed*	*tmed*	*omed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*lmed*	*mean*
		2.024	2.072	2.073	2.535	2.831	2.832	3.113	3.113	3.143	3.282
		cmed	*tmed*	*omed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.973	0.993	0.997	1.047	1.076	1.803	1.803	2.103	2.103	2.355
	200	*cmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	*tmed*	*mean*
		2.166	2.214	2.723	3.191	3.214	3.268	3.684	3.685	3.828	4.441
		cmed	*lmed*	*omed*	*tmed*	*vmed*	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		1.414	1.423	1.448	1.549	1.619	2.667	2.669	3.057	3.086	3.928
$MN_{12}(.20)$	20	*cmed*	*tmed*	*omed*	*vmed*	*mean*	*lmed*	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		2.047	2.122	2.148	2.467	3.437	3.487	3.537	3.537	3.657	3.657
		cmed	*omed*	*tmed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.967	1.004	1.005	1.072	1.111	2.202	2.202	2.331	2.331	2.465
	60	*cmed*	*omed*	*tmed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		2.549	2.696	2.706	3.167	3.760	4.524	4.533	4.928	4.928	5.171
		lmed	*cmed*	*tmed*	*omed*	*vmed*	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.612	1.633	1.691	1.715	1.837	3.482	3.492	3.911	3.911	4.287
	200	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		4.196	4.434	4.447	5.349	6.015	7.110	7.223	8.120	8.244	9.168
		cmed	*lmed*	*tmed*	*omed*	*vmed*	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		3.074	3.083	3.181	3.215	3.493	5.952	6.073	6.803	6.828	7.916

TABLE 1. (Continued) Estimated accuracy as measured by M and B. For each distribution and each sample size, estimators ordered according to increasing M (first and second lines) and B (third and fourth lines).

$MN_{21}(.05)$	20	cmed	omed	tmed	vmed	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean	lmed
		1.450	1.470	1.503	1.708	2.278	2.278	2.304	2.304	2.305	2.480
		cmed	tmed	omed	lmed	vmed	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		0.370	0.391	0.393	0.396	0.410	1.256	1.256	1.365	1.365	1.499
	60	cmed	omed	tmed	vmed	lmed	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		1.587	1.619	1.634	1.984	2.536	2.807	2.819	3.190	3.190	3.355
		cmed	tmed	omed	lmed	vmed	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		0.620	0.625	0.637	0.640	0.689	1.794	1.796	2.241	2.241	2.683
	200	cmed	omed	tmed	vmed	lmed	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	mean
		2.029	2.078	2.136	2.478	3.178	3.710	3.722	4.600	4.613	5.681
		cmed	omed	tmed	lmed	vmed	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		1.186	1.211	1.228	1.234	1.300	2.932	3.025	3.820	3.823	5.027
$MN_{21}(.10)$	20	cmed	tmed	omed	vmed	lmed	mean	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$
		1.828	1.921	1.981	2.216	3.130	3.977	4.067	4.067	4.103	4.103
		cmed	tmed	omed	lmed	vmed	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		0.782	0.827	0.832	0.877	0.895	2.840	2.840	3.034	3.034	3.180
	60	cmed	tmed	omed	vmed	lmed	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		2.127	2.244	2.273	2.589	3.200	4.965	4.975	5.473	5.473	5.762
		cmed	lmed	tmed	omed	vmed	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		1.308	1.337	1.360	1.389	1.458	4.209	4.217	4.834	4.834	5.452
	200	cmed	omed	tmed	vmed	lmed	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	mean
		3.222	3.451	4.109	4.125	4.385	7.842	7.849	9.146	9.247	10.22
		lmed	cmed	omed	tmed	vmed	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	mean
		2.431	2.432	2.570	2.678	2.767	7.005	7.217	8.450	8.483	9.851
$MN_{21}(.20)$	20	cmed	omed	tmed	vmed	lmed	mean	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		3.195	3.939	4.050	4.278	5.298	7.496	7.627	7.627	7.734	7.734
		cmed	vmed	lmed	omed	tmed	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		1.794	2.068	2.100	2.163	2.184	5.609	5.609	5.950	5.950	6.127
	60	cmed	omed	tmed	vmed	lmed	$hd_{.10}$	$sd_{.10}$	mean	$hd_{.05}$	$sd_{.05}$
		4.345	5.038	5.427	5.594	6.588	11.15	11.15	11.81	11.91	11.91
		cmed	lmed	vmed	tmed	omed	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		3.115	3.424	3.462	3.520	3.546	9.481	9.486	10.11	10.11	10.72
	200	cmed	omed	tmed	vmed	lmed	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	mean
		6.980	8.174	8.327	9.228	9.872	18.71	19.07	19.87	19.97	20.64
		cmed	lmed	vmed	omed	tmed	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	mean
		5.594	5.967	6.262	6.368	6.400	17.32	17.44	18.59	18.62	19.94
$MN_{22}(.05)$	20	cmed	omed	tmed	vmed	lmed	mean	$hd_{.05}$	$sd_{.05}$	$hd_{.10}$	$sd_{.10}$
		1.460	1.533	1.534	1.722	2.458	2.627	2.702	2.702	2.713	2.713
		lmed	cmed	tmed	omed	vmed	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		0.326	0.328	0.329	0.329	0.383	1.318	1.318	1.407	1.407	1.538
	60	cmed	tmed	omed	vmed	$hd_{.10}$	$sd_{.10}$	lmed	$hd_{.05}$	$sd_{.05}$	mean
		1.589	1.632	1.640	1.914	2.460	2.460	2.818	2.967	2.967	3.203
		cmed	lmed	tmed	omed	vmed	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	mean
		0.609	0.615	0.622	0.635	0.707	1.670	1.671	2.148	2.148	2.618
	200	cmed	omed	vmed	lmed	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	mean	tmed
		1.882	1.911	2.243	2.924	3.286	3.350	4.188	4.193	5.632	7.983
		lmed	cmed	omed	vmed	tmed	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	mean
		0.986	1.056	1.082	1.177	1.399	2.643	2.662	3.484	3.504	4.940

Table 1. (Continued) Estimated accuracy as measured by M and B. For each distribution and each sample size, estimators ordered according to increasing M (first and second lines) and B (third and fourth lines).

$MN_{22}(.10)$	20	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.794	1.796	1.871	2.152	2.835	4.115	4.115	4.182	4.182	4.187
		tmed	*cmed*	*omed*	*lmed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		0.732	0.735	0.762	0.818	0.842	2.755	2.755	2.965	2.965	3.140
	60	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		2.200	2.275	2.363	2.689	3.383	5.121	5.135	5.959	5.959	6.189
		cmed	*lmed*	*tmed*	*omed*	*vmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.265	1.277	1.287	1.352	1.426	3.987	3.992	4.781	4.781	5.511
	200	*cmed*	*omed*	*vmed*	*tmed*	*lmed*	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		3.250	3.508	3.950	4.076	4.618	7.508	7.751	9.274	9.339	10.92
		cmed	*lmed*	*omed*	*tmed*	*vmed*	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		2.378	2.434	2.508	2.562	2.723	6.560	6.728	8.108	8.143	10.00
$MN_{22}(.20)$	20	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.05}$	$sd_{.05}$	*mean*	$hd_{.10}$	$sd_{.10}$
		3.040	3.507	3.519	3.677	5.230	8.296	8.296	8.372	8.508	8.508
		cmed	*tmed*	*vmed*	*omed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		1.755	1.993	1.997	2.023	2.115	5.691	5.691	6.022	6.022	6.346
	60	*cmed*	*tmed*	*omed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		4.486	4.863	4.946	5.935	6.480	12.23	12.25	13.44	13.44	13.89
		cmed	*lmed*	*tmed*	*omed*	*vmed*	$sd_{.10}$	$hd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		3.168	3.379	3.387	3.524	3.535	9.592	9.596	10.51	10.51	11.24
	200	*cmed*	*omed*	*vmed*	*tmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	*mean*
		6.807	7.727	9.105	9.579	9.899	19.09	19.21	20.72	20.83	22.45
		cmed	*lmed*	*omed*	*vmed*	*tmed*	$sd_{.10}$	$hd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
		5.407	5.656	6.044	6.089	6.102	16.20	16.48	18.11	18.13	20.01

TABLE 2. Averaged midrank of each estimator with respect to M over all three sample sizes and each distribution.

	$cmed$	$omed$	$tmed$	$vmed$	$hd_{.10}$	$sd_{.10}$	$hd_{.05}$	$sd_{.05}$	$lmed$	$mean$
$N_2(0,I)$	6.33	7.00	7.67	9.00	3.50	4.83	2.67	3.00	10.00	1.00
$DE_2(0,I)$	3.33	2.33	3.33	1.00	6.50	7.17	8.00	7.67	8.67	7.00
UD	6.33	7.00	7.67	9.00	4.67	4.33	2.50	2.50	10.00	1.00
t_3	1.00	2.67	3.33	3.67	6.50	6.50	7.67	8.00	7.00	8.67
$C_2(0,I)$	1.00	2.33	3.33	3.67	6.50	6.50	8.67	9.00	4.67	9.33
$MN(.05)$	2.67	3.67	4.33	8.00	5.17	5.50	5.33	4.67	10.00	5.67
$MN(.10))$	1.33	2.67	4.67	7.00	4.50	5.17	6.67	6.33	10.00	6.67
$MN(.20))$	1.00	2.33	3.67	5.00	6.17	5.50	7.67	7.33	9.33	7.00
$MNC(.05)$	2.00	3.67	4.33	5.00	5.50	6.50	6.00	6.33	7.00	8.67
$MNC(.10))$	4.00	1.67	4.00	4.67	4.83	5.50	7.00	7.33	6.67	9.33
$MNC(.20))$	1.00	2.67	6.00	4.00	4.83	5.17	8.33	8.00	5.67	9.33
$MNU(.05)$	1.67	3.00	4.33	5.00	4.83	6.17	6.67	7.67	6.33	9.33
$MNU(.10))$	1.33	2.67	4.00	4.00	6.17	6.83	6.67	7.33	6.00	10.00
$MNU(.20))$	1.00	2.33	5.33	3.33	5.50	6.17	8.33	8.67	5.00	9.33
$MN_{11}(.05)$	2.00	4.00	3.67	8.67	4.83	5.50	5.33	5.67	10.00	5.33
$MN_{11}(.10))$	1.00	2.00	3.67	3.67	6.17	6.83	7.33	7.67	9.67	7.00
$MN_{11}(.20))$	1.00	2.33	2.67	4.00	6.83	6.83	8.17	8.17	7.67	7.33
$MN_{12}(.05)$	1.00	2.00	4.67	6.00	5.17	5.83	6.67	7.00	10.00	6.67
$MN_{12}(.10))$	1.33	2.67	4.00	3.67	6.17	6.83	7.33	7.00	7.67	8.33
$MN_{12}(.20))$	1.00	2.67	2.33	4.00	7.33	7.67	8.33	8.00	5.33	8.33
$MN_{21}(.05)$	1.00	2.00	3.00	4.00	6.17	6.17	8.33	8.00	6.67	9.67
$MN_{21}(.10))$	1.00	2.67	2.33	4.00	7.17	6.50	9.00	8.67	5.00	8.67
$MN_{21}(.20))$	1.00	2.00	3.00	4.00	7.33	7.67	8.50	8.50	5.00	8.00
$MN_{22}(.05)$	1.00	2.33	5.00	3.67	6.83	6.83	8.00	7.67	5.33	8.33
$MN_{22}(.10))$	1.00	2.67	2.67	3.67	6.33	6.67	8.50	8.50	5.00	10.00
$MN_{22}(.20))$	1.00	2.67	2.67	3.67	7.33	7.67	7.83	7.83	5.00	9.33
$\overline{r}_M$	1.82	2.92	4.06	4.82	5.88	6.26	7.13	7.17	7.26	7.67

TABLE 3. Averaged midrank of each estimator with respect to B over all three sample sizes and each distribution.

	cmed	*omed*	*tmed*	*vmed*	*lmed*	$hd_{.10}$	$sd_{.10}$	$sd_{.05}$	$hd_{.05}$	*mean*
$N_2(0,I)$	7.00	6.67	4.33	7.33	7.00	5.33	6.00	4.83	4.83	1.67
$DE_2(0,I)$	3.33	2.67	5.00	3.00	4.00	8.50	8.17	4.83	7.17	8.33
UD	6.67	5.67	6.00	8.33	7.00	6.00	6.33	3.67	4.00	1.33
t_3	4.67	4.33	2.67	5.00	7.33	5.17	5.83	5.00	5.67	9.33
$C_2(0,I)$	2.67	2.00	4.00	2.00	4.67	6.83	8.17	8.00	7.33	9.33
$MN(.05)$	5.67	5.33	3.00	3.67	4.67	5.50	7.17	6.50	8.83	4.67
$MN(.10)$	3.00	4.33	4.00	2.67	7.33	7.83	6.17	7.67	6.67	5.33
$MN(.20)$	3.67	4.00	6.33	6.33	6.33	2.83	3.17	7.83	6.83	7.67
$MNC(.05)$	4.67	5.00	5.33	2.67	6.00	5.50	4.83	5.67	6.67	8.67
$MNC(.10)$	6.67	2.33	2.67	3.67	5.00	4.83	5.17	8.00	7.33	9.33
$MNC(.20)$	2.00	3.00	5.67	2.00	5.33	6.50	5.17	7.33	8.67	9.33
$MNU(.05)$	2.33	2.33	3.67	3.00	4.33	5.83	7.83	8.67	8.33	8.67
$MNU(.10)$	5.33	3.67	4.33	1.00	7.00	5.83	5.17	7.33	6.67	8.67
$MNU(.20)$	3.00	2.33	6.67	2.33	4.67	5.17	5.83	8.33	7.67	9.00
$MN_{11}(.05)$	1.67	3.00	2.33	4.67	3.33	6.17	6.83	8.33	8.67	10.00
$MN_{11}(.10)$	1.00	2.33	3.00	5.00	3.67	6.50	6.50	8.33	8.67	10.00
$MN_{11}(.20)$	1.00	3.67	2.67	5.00	2.67	6.83	6.17	8.33	8.67	10.00
$MN_{12}(.05)$	1.67	2.67	3.67	3.67	3.33	6.50	6.50	8.33	8.67	10.00
$MN_{12}(.10)$	1.00	3.00	2.67	5.00	3.33	6.50	6.50	8.33	8.67	10.00
$MN_{12}(.20)$	1.33	3.33	3.00	5.00	2.33	6.83	6.17	8.33	8.67	10.00
$MN_{21}(.05)$	1.00	2.67	2.33	5.00	4.00	6.50	6.50	8.67	8.33	10.00
$MN_{21}(.10)$	1.33	3.33	3.00	5.00	2.33	6.83	6.17	8.33	8.67	10.00
$MN_{21}(.20)$	1.00	4.33	4.67	2.67	2.33	6.83	6.17	8.33	8.67	10.00
$MN_{22}(.05)$	1.67	3.33	4.00	4.67	1.33	6.17	6.83	8.33	8.67	10.00
$MN_{22}(.10)$	1.33	3.33	2.67	5.00	2.67	6.50	6.50	8.33	8.67	10.00
$MN_{22}(.20)$	1.00	3.67	3.33	4.00	3.00	6.83	6.17	8.33	8.67	10.00
$\overline{r}_B$	2.91	3.55	3.88	4.14	4.42	6.18	6.23	7.46	7.71	8.51

TABLE 4. Scores of robustness with respect to sample size and M.

n	*cmed*	*omed*	*tmed*	*vmed*	$hd_{.1}$	$sd_{.1}$	*lmed*	$hd_{.05}$	$sd_{.05}$	*mean*
20	1.27	2.95	2.45	4.41	7.95	7.95	6.95	7.39	7.34	6.32
60	1.05	2.27	3.09	4.77	5.73	5.55	6.91	8.39	8.43	8.82
200	1.82	2.55	5.86	4.82	4.27	5.59	7.00	6.68	6.77	9.64
$\overline{s}$	1.38	2.59	3.80	4.67	5.98	6.36	6.95	7.48	7.52	8.26

TABLE 5. Scores of robustness with respect to sample size and B.

n	$cmed$	$omed$	$tmed$	$vmed$	$lmed$	$hd_{.1}$	$sd_{.1}$	$sd_{.05}$	$hd_{.05}$	$mean$
20	2.32	3.14	3.36	3.95	3.86	6.82	6.82	8.20	8.16	8.36
60	2.32	3.45	3.45	3.68	4.00	5.64	5.68	8.57	8.61	9.59
200	2.73	3.36	4.50	3.82	4.36	6.05	6.00	7.18	7.59	9.41
$\overline{s}$	2.45	3.32	3.77	3.82	4.08	6.17	6.17	7.98	8.12	9.12